VARIATION OF THE ELECTRICAL CONDUCTIVITY OF AN IONIZED GAS IN THE INITIAL SECTION OF A PLANE DUCT

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§1. Formulation of the problem. The variation of the electrical conductivity of an ionized gas in the initial section of a duct is studied in relation to the following problem: In a plane semi-infinite duct ($x \geq 0$, $|y| < a$) a gas is moving with constant velocity v and temperature T_0. At time t = 0 in the inlet section (x = 0) we are given the concentration of an easily ionized seed element $n = n_0 f(t)$ and the gas temperature $T = T_0 g(t)$. At t = 0 the wall temperature is assumed to be T_0 and the seed concentration zero. The temperature distribution T (x, y, t) and the seed concentration n (x, y, t) are investigated as functions of the coordinates and time and in the region of the duct satisfy the approximate equations [1-3]

$$\frac{\partial n}{\partial t}+v\frac{\partial n}{\partial x}=D\frac{\partial^2 n}{\partial y^2},\qquad \frac{\partial T}{\partial t}+v\frac{\partial T}{\partial x}=\frac{\lambda}{\rho c_p}\frac{\partial^2 T}{\partial y^2}. \tag{1.1}$$

In the duct wall region the temperature satisfies the thermal conductivity equation

$$\frac{\partial T_w}{\partial t}=\frac{\lambda_w}{\rho_w c_w}\frac{\partial^2 T_w}{\partial y^2}. \tag{1.2}$$

The initial and boundary conditions of the problem are

$$\begin{gathered} n=0,\quad T=T_0,\quad T_w=T_0\quad \text{at } t=0,\\ n=n_0 f(t),\quad T=T_0 g(t)\quad \text{at } x=0,\\ n=0,\quad T=T_w,\quad \lambda\partial T/\partial y=\lambda_w\partial T_w/\partial y\quad \text{at } |y|=a. \end{gathered} \tag{1.3}$$

Going over to dimensionless variables

$$\beta=\frac{n}{n_0},\quad \theta=\frac{T-T_0}{T_0},\quad \tau=\frac{vt}{a},\quad \xi=\frac{x}{a},\quad \eta=\frac{y}{a}, \tag{1.4}$$

we obtain

$$\frac{\partial\beta}{\partial\tau}+\frac{\partial\beta}{\partial\xi}=\frac{1}{\gamma}\frac{\partial^2\beta}{\partial\eta^2},\qquad \frac{\partial\theta}{\partial\tau}+\frac{\partial\theta}{\partial\xi}=\frac{1}{\delta}\frac{\partial^2\theta}{\partial\eta^2}\quad (|\eta|<1), \tag{1.5}$$

$$\frac{\partial\theta_w}{\partial\tau}=\frac{1}{\delta_w}\frac{\partial^2\theta_w}{\partial\eta^2}\qquad (|\eta|>1), \tag{1.6}$$

$$\beta=\theta=\theta_w=0\qquad \text{at } \tau=0, \tag{1.7}$$

$$\beta=f(\tau),\quad \theta=g(\tau)-1\qquad \text{at } \xi=0, \tag{1.8}$$

$$\beta=0,\quad \theta=\theta_w,\quad \frac{\partial\theta}{\partial\eta}=\varkappa\frac{\partial\theta_w}{\partial\eta}\qquad \text{at } |\eta|=1. \tag{1.9}$$

Here

$$\gamma=\frac{av}{D},\quad \delta=\frac{\rho c_p av}{\lambda},\quad \delta_w=\frac{\rho_w c_w av}{\lambda_w},\quad \varkappa=\frac{\lambda_w}{\lambda}. \tag{1.10}$$

If the quantities [$\beta(\xi,\eta,\tau)$ and $\theta(\xi,\eta,\tau)$] are found, the electrical conductivity of the gas can be calculated as follows: for a mixture of the basic inert gas and a small quantity of alkali seed vapor the equilibrium electron concentration can be defined by Saha's equation [4, 5]

$$\frac{n_e^2}{n-n_e}=\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2}\exp\left(-\frac{e\varphi}{kT}\right) \tag{1.11}$$

where n is the seed concentration and φ the seed ionization potential. For a small degree of seed ionization $n_e \ll n$, we have

$$n_e=\sqrt{n}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/4}\exp\left(-\frac{e\varphi}{2kT}\right). \tag{1.12}$$

The electrical conductivity can be defined by the well-known formula

$$\sigma=\frac{n_e e^2\tau_e}{m_e}\qquad \left(\tau_e=\frac{l_e}{v_e},\quad v_e=\frac{\sqrt{8kT}}{\sqrt{\pi m_e}}\right),$$

in which τ_e is the mean free time between electron collisions with heavy particles, l_e the mean free path of the electron, and v_e the thermal velocity of the electron. Substituting β and θ for n_e and T with $\sigma_0-\sigma$ at $\xi = 0$ and writing $\sigma^\circ = \sigma/\sigma_0$, we obtain for the electrical conductivity in dimensionless form

$$\sigma^\circ=\left(\frac{\beta}{f}\right)^{1/2}\left(\frac{1+\theta}{g}\right)^{1/4}\exp\left[\frac{e\varphi}{2kT_0}\left(\frac{1}{g}-\frac{1}{1+\theta}\right)\right]. \tag{1.13}$$

§2. General solution of the problem and some particular cases. Successively applying the Laplace transformations

$$F_{(\tau)}=\int_0^\infty e^{-p\tau}f d\tau,\qquad F_{(\xi)}=\int_0^\infty e^{-s\xi}f d\xi \tag{2.1}$$

with respect to the variables τ and ξ, we get

$$\begin{gathered} \frac{d^2\beta(\tau,\xi)}{d\eta^2}=\gamma(p+s)\beta(\tau,\xi)-\gamma f(\tau),\\ \frac{d^2\theta(\tau,\xi)}{d\eta^2}=\delta(p+s)\theta(\tau,\xi)+\delta\left(\frac{1}{p}-g(\tau)\right),\\ \frac{d^2\theta_w(\tau,\xi)}{d\eta^2}=\delta_w p\theta_w(\tau,\xi), \end{gathered} \tag{2.2}$$

and the boundary conditions for [$\eta = \pm 1$]

$$\beta(\tau,\xi)=0,\qquad \theta(\tau,\xi)=\theta_w(\tau,\xi),\qquad \frac{d\theta(\tau,\xi)}{d\eta}=\varkappa\frac{d\theta_w(\tau,\xi)}{d\eta}. \tag{2.3}$$

Omitting the intermediate calculation we arrive at the final expressions

$$\beta(\tau,\xi)=\frac{f(\tau)}{p+s}\left[1-\frac{\operatorname{ch}\eta\sqrt{\gamma(p+s)}}{\operatorname{ch}\sqrt{\gamma(p+s)}}\right], \tag{2.4}$$

$$\theta(\tau,\xi)=\frac{g(\tau)-p^{-1}}{p+s}\times\left[1-\frac{\operatorname{ch}\eta\sqrt{\delta(p+s)}}{\operatorname{ch}\sqrt{\delta(p+s)}+\varkappa^{-1}\sqrt{(\delta/\delta_w)(p+s)/p}\operatorname{sh}\sqrt{\delta(p+s)}}\right]. \tag{2.5}$$

Further calculations will be made for constant values of [$f(\tau) = 1$ and $g(\tau) = m$], because the case of arbitrary dependence on time can be obtained from this expression by means of the Duhamel integral. When $\varkappa \to \infty$ the function β is proportional to the function θ, so it is sufficient to substitute in expression (2.5). Setting $q = \delta(p + s)$, we obtain

$$\theta(\tau,\xi)=\frac{\delta(m-1)}{pq}\left[1-\frac{\operatorname{ch}\eta\sqrt{q}}{\operatorname{ch}\sqrt{q}+\alpha\sqrt{q/p}\operatorname{sh}\sqrt{q}}\right],\qquad \alpha=\frac{1}{\varkappa\sqrt{\delta_w}}. \tag{2.6}$$

Application of the Riemann-Mellin formula gives

$$\theta=\frac{m-1}{2\pi i}\int_{\sigma_1-i\infty}^{\sigma_1+i\infty}\exp\frac{q\xi}{\delta}\frac{dq}{q}\frac{1}{2\pi i}\int_{\sigma_2-i\infty}^{\sigma_2+i\infty}e^{p(\tau-\xi)}\times\left[\frac{1}{p}-\frac{1}{\sqrt{p}(\sqrt{p}+\mu)}\frac{\operatorname{ch}\eta\sqrt{q}}{\operatorname{ch}\sqrt{q}}\right]dp,\quad \mu=\alpha\sqrt{q}\operatorname{th}\sqrt{q}. \tag{2.7}$$

Evaluating the inner integral by means of the formula

$$\frac{1}{2\pi i}\int_{\sigma_1-i\infty}^{\sigma_1+i\infty}\frac{e^{pu}dp}{\sqrt{p}(\sqrt{p}+\mu)]}=e^{\mu^2 u}[1-\Phi(\mu\sqrt{u})] \qquad (u>0) \tag{2.8}$$

in which $\Phi(z)$ is the probability function, we obtain a solution in the form of a single integral

$$\theta=0, \qquad \tau<\xi$$

$$\theta=\frac{m-1}{2\pi i}\int_{\sigma_1-i\infty}^{\sigma_1+i\infty}\exp\frac{q\xi}{\delta}\left\{1-\frac{\operatorname{ch}\eta\sqrt{q}}{\operatorname{ch}\sqrt{q}}e^{\mu^2 u}[1-\Phi(\mu\sqrt{u})]\right\}\frac{dq}{q},$$

$$\tau>\xi \tag{2.9}$$

where we have set $u = \tau - \xi$.

The result obtained shows, above all, that a wave whose shape depends on the variables ξ and $\tau - \xi$ is propagated along the duct with a velocity v. Conversion of the integral (2.9) to a form suitable for evaluation is rather complicated. For example, using summation of the residues at the essential singular points $q_n = -{}^1/_4(2n+1)^2\pi^2$ the solution can be given as a double series. Without giving these results, we shall show the possibility of obtaining an approximate solution for small or large values of the parameter $\varkappa = \lambda_W/\lambda$. For $\varkappa \to \infty$ ($\alpha \to 0$) one can use the asymptotic expression $\Phi(z) \approx 2\pi^{-1/2}z$ ($|z| \to 0$) and obtain the solution in the form

$$\frac{\theta}{m-1}\Big|_{\tau>\xi}=\frac{1}{2\pi i}\int_{\sigma_1-i\infty}^{\sigma_1+i\infty}\exp\frac{q\xi}{\delta}\left[1-\frac{\operatorname{ch}\eta\sqrt{q}}{\operatorname{ch}\sqrt{q}}\right]\frac{dq}{q}+$$

$$+\frac{2\alpha\sqrt{\tau-\xi}}{\sqrt{\pi}}\frac{1}{2\pi i}\int_{\sigma_1-i\infty}^{\sigma_1+i\infty}\exp\frac{q\xi}{\delta}\frac{\operatorname{ch}\eta\sqrt{q}\operatorname{sh}\sqrt{q}}{\operatorname{ch}^2\sqrt{q}}\frac{dq}{\sqrt{q}}. \tag{2.10}$$

In particular, for $\alpha = 0$, corresponding to a constant duct wall temperature, we find by means of the residue theorem

$$\theta|_{\tau>\xi}=\frac{4}{\pi}(m-1)\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\cos\frac{2k+1}{2}\pi\eta\exp\times$$

$$\times\left[-\frac{\xi}{\delta}\left(\frac{2k+1}{2}\pi\right)^2\right]. \tag{2.11}$$

In the other boundary case ($\varkappa = 0$) of a thermally insulated duct wall, we obtain

$$\theta=m-1 \quad \text{at } \xi<\tau, \qquad \theta=0 \quad \text{at } \xi>\tau. \tag{2.12}$$

Thus, in both cases when $\xi < \tau$ the temperature is not dependent on time. The function β, on the basis of (2.4) is in general independent of the parameter $\varkappa$ and $\beta = 0$ when $\xi > \tau$, but when $\xi < \tau$ we obtain an expression (2.11) for β in which the parameter δ should be replaced by γ and the factor (m − 1) by unity. Limiting ourselves to the first term in (2.11), we obtain the following approximate expressions for the length ξ' of the thermal inlet section ($\varkappa = \infty$) and for the length ξ'' of the diffusion inlet section:

$$\xi'=4\delta/\pi^2, \qquad \xi''=4\gamma/\pi^2. \tag{2.13}$$

For thermally insulated walls it follows from (2.12) that there is no inlet section. It should be noted that the ratio of the parameters δ and γ is the Lewis number

$$L=\delta/\gamma=\rho c_p D/\lambda. \tag{2.14}$$

Usually $L \approx 1$, so that ξ' and ξ'' are of the same order. The parameter δ for typical conditions in the duct of a MHD generator (e.g., for argon with p = 1 atm, T = 3000° K, a = 1 m, v = 10^3 m/sec) is of the order of 10^6, i.e., the lengths of the thermal and diffusion inlet sections obtained are considerably greater than any reasonable duct length. It follows that under these conditions we can neglect the reduction of gas temperature and seed concentration along the duct due to the thermal and diffusion flows to the wall. Consequently, the physical properties of the walls have practically no effect on the behavior of the electrical conductivity in the flow core.

In conclusion, we shall consider the case of a steady oscillatory regime when $f(\tau) = 1$, $g(\tau) = 1 + \nu\sin\omega\tau$, i.e.,

$$f^{(\tau)}=\frac{1}{p}, \qquad g^{(\tau)}=\frac{1}{p}+\frac{\nu\omega}{\omega^2+p^2}.$$

In this case the concentration β is given by

$$\beta=\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\cos\frac{2k+1}{2}\pi\eta\exp\left[-\frac{\xi}{\gamma}\left(\frac{2k+1}{2}\pi\right)^2\right] \tag{2.15}$$

and to find θ we must, using (2.5), calculate the residues at the poles $p = \pm i\omega$. For $\varkappa = \infty$ we obtain as a result of calculation

$$\theta=\frac{4\nu}{\pi}\sum_{k=0}^{\infty}(-1)^k\frac{\cos{}^1/_2(2k+1)\pi\eta\sin\omega(\tau-\xi)}{2k+1}\exp\times$$

$$\times\left[-\frac{\xi}{\delta}\left(\frac{2k+1}{2}\pi\right)^2\right]. \tag{2.16}$$

i.e., the temperature is propagated along the duct in the form of a damped wave with velocity v, the decrement $\Delta = 4\delta/\pi^2$ not being dependent on frequency. For $\varkappa = 0$ the temperature variation is propagated in the form of an undamped wave

$$\theta=\nu\sin\omega(\tau-\xi). \tag{2.17}$$

The influence of temperature fluctuations on the variation of the electrical conductivity of the flow can be easily determined by means of (1.13).

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